Topological properties of regular generalized function algebras

H. Vernaeve*

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Abstract

We investigate density of various subalgebras of regular generalized functions in the special Colombeau algebra $\mathcal{G}(\Omega)$ of generalized functions.

1 Introduction

M. Oberguggenberger introduced the algebra $\mathcal{G}^\infty(\Omega)$ of regular generalized functions in order to develop a hypoelliptic regularity theory and hyperbolic propagation of singularities in the algebra $\mathcal{G}(\Omega)$ of Colombeau generalized functions [13], where it takes over the role of the subalgebra of $\mathcal{C}^\infty$-regular functions in the space $\mathcal{D}'(\Omega)$ of distributions. It thus became the starting point of investigations of microlocal regularity in generalized function algebras (see [5, 7, 9, 10, 12, 16] and the references therein). More recently, various other subalgebras of regular generalized functions have been considered, from the point of view of generalized analytic functions [1], kernel theorems [3], propagation of singularities [14] and microlocal analysis [4]. We show that, in contrast with the situation of $\mathcal{C}^\infty(\Omega)$ as a subalgebra of $\mathcal{D}'(\Omega)$ (and therefore maybe surprisingly), the subalgebra $\mathcal{G}^\infty(\Omega)$ and the subalgebras $\mathcal{G}_{\mathcal{L}_a}(\Omega)$ considered in [3, 4] are not dense in the algebra $\mathcal{G}(\Omega)$. On the other hand, the subalgebra of sublinear or S-analytic generalized functions is dense in $\mathcal{G}(\Omega)$.

2 Notations

Let $\Omega \subseteq \mathbb{R}^d$ be open. By $K \subset\subset \Omega$, we denote a compact subset of Ω.
For $u \in \mathcal{C}^\infty(\Omega)$, $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}^d$, let $p_{\alpha,K}(u) := \sup_{x\in K} |\partial^\alpha u(x)|$. For $k \in \mathbb{N}$, let $p_{k,K}(u) := \max_{|\alpha|=k} p_{\alpha,K}(u)$.
The special algebra of Colombeau generalized functions (see e.g. [8]) is $\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\begin{aligned}
\mathcal{E}_M(\Omega) = \big\{&(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1)} : (\forall K \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d)(\exists N \in \mathbb{N})\\
&(p_{\alpha,K}(u_\varepsilon) \le \varepsilon^{-N}, \text{ for small } \varepsilon)\big\}\\
\mathcal{N}(\Omega) = \big\{&(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1)} : (\forall K \subset\subset \Omega)(\forall \alpha \in \mathbb{N}^d)(\forall m \in \mathbb{N})\\
&(p_{\alpha,K}(u_\varepsilon) \le \varepsilon^{m}, \text{ for small } \varepsilon)\big\}.
\end{aligned}$$

*Dept. Of Mathematics, Ghent University. E-mail: `hvernaev@cage.ugent.be`

By $[(u_\varepsilon)_\varepsilon]$, we denote the generalized function with representative $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$.
The subalgebra $\mathcal{G}_c(\Omega)$ of compactly supported generalized functions consists of those $u \in \mathcal{G}(\Omega)$ such that for some $K \subset\subset \Omega$, the restriction of u to $\Omega \setminus K$ equals 0 (as an element of $\mathcal{G}(\Omega \setminus K)$).
For $K \subset\subset \Omega$, the algebra $\mathcal{G}^\infty(K)$ consists of those $u \in \mathcal{G}(\Omega)$ such that for one (and hence for each) representative $(u_\varepsilon)_\varepsilon$,

$$(\exists N \in \mathbb{N})(\forall \alpha \in \mathbb{N}^d)\big(p_{\alpha,K}(u_\varepsilon) \le \varepsilon^{-N}, \text{ for small } \varepsilon\big).$$

For $(z_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1)}$, the valuation $\mathrm{v}(z_\varepsilon) := \sup\{b \in \mathbb{R} : |z_\varepsilon| \le \varepsilon^b, \text{ for small } \varepsilon\}$ and the so-called sharp norm $|z_\varepsilon|_{\mathrm{e}} := e^{-\mathrm{v}(z_\varepsilon)}$. For $u \in \mathcal{G}(\Omega)$, $P_{\alpha,K}(u) := |p_{\alpha,K}(u_\varepsilon)|_{\mathrm{e}}$ ($\alpha \in \mathbb{N}^d$) and $P_{k,K}(u) := |p_{k,K}(u_\varepsilon)|_{\mathrm{e}}$ ($k \in \mathbb{N}$), independent of the representative $(u_\varepsilon)_\varepsilon$ of u. The ultra-pseudo-seminorms $P_{\alpha,K}$ ($\alpha \in \mathbb{N}^d$, $K \subset\subset \Omega$) determine a topology on $\mathcal{G}(\Omega)$ called sharp topology [2, 6, 16]. Then $u \in \mathcal{G}^\infty(K)$ iff $\sup_{k\in\mathbb{N}} P_{k,K}(u) < +\infty$. Further, the algebra $\mathcal{G}^\infty(\Omega) := \bigcap_{K\subset\subset\Omega} \mathcal{G}^\infty(K)$ [13].
For $K \subset\subset \Omega$, the algebra $\mathcal{G}_{\mathcal{L}_a}(K)$ of generalized functions of sublinear growth with slope smaller than $a > 0$ ($a \in \mathbb{R}$) on K consists of those $u \in \mathcal{G}(\Omega)$ such that for one (and hence for each) representative $(u_\varepsilon)_\varepsilon$,

$$(\exists a' < a)(\exists b \in \mathbb{R})(\forall \alpha \in \mathbb{N}^d)(p_{\alpha,K}(u_\varepsilon) \le \varepsilon^{-a'|\alpha|-b}, \text{ for small } \varepsilon)$$

or, equivalently,

$$(\exists a' < a)(\exists c \in \mathbb{R})(P_{\alpha,K}(u) \le c e^{a'|\alpha|}, \forall \alpha \in \mathbb{N}^d),$$

which can still be expressed concisely by $\limsup_{k\to\infty} \frac{\ln P_{k,K}(u)}{k} < a$. Since $P_{\alpha,K}(uv) \le \max_{\beta\le\alpha}(P_{\beta,K}(u)P_{\alpha-\beta,K}(v))$ by Leibniz's rule, $\mathcal{G}_{\mathcal{L}_a}(K)$ are subalgebras of $\mathcal{G}(\Omega)$.
For $a = 0$, $\mathcal{G}_{\mathcal{L}_0}(K) := \bigcap_{a>0} \mathcal{G}_{\mathcal{L}_a}(K)$. Clearly, $\mathcal{G}^\infty(K) \subseteq \mathcal{G}_{\mathcal{L}_0}(K)$.
Again, the algebras $\mathcal{G}_{\mathcal{L}_a}(\Omega) := \bigcap_{K\subset\subset\Omega} \mathcal{G}_{\mathcal{L}_a}(K)$ ($a \ge 0$) [3, 4]. Clearly, $\mathcal{G}^\infty(\Omega) \subseteq \mathcal{G}_{\mathcal{L}_0}(\Omega)$.
By definition, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ is sublinear [1, 15] iff for each $K \subset\subset \Omega$ and each $(x_\varepsilon)_\varepsilon \in K^{(0,1)}$, there exists $k \in \mathbb{R}$ and $(p_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\lim_{n\to\infty} p_n + kn = \infty$ and for each $\alpha \in \mathbb{N}^d$, $|\partial^\alpha u_\varepsilon(x_\varepsilon)| \le \varepsilon^{p_{|\alpha|}}$, for small ε. It can be shown [1, Thm. 5.7], [15, Thm. 10] that the algebra of sublinear generalized functions exactly contains those $u \in \mathcal{G}(\Omega)$ satisfying a natural condition of analyticity (called S-real analyticity in [15]).
Sublinearity can still be characterized as follows by means of the algebras $\mathcal{G}_{\mathcal{L}_a}(K)$:

Lemma 2.1. *Let $u \in \mathcal{G}(\Omega)$. Then u is sublinear iff for each $K \subset\subset \Omega$, there exists $a > 0$ ($a \in \mathbb{R}$) such that $u \in \mathcal{G}_{\mathcal{L}_a}(K)$.*

Proof. $\Rightarrow$: let u be sublinear and suppose that there exists $K \subset\subset \Omega$ such that $u \notin \mathcal{G}_{\mathcal{L}_a}(K)$, for each $a > 0$. Then we find $\alpha_n \in \mathbb{N}$ (for each $n \in \mathbb{N}$), $\varepsilon_{n,m} \in (0, 1/m)$ (for each $n, m \in \mathbb{N}$) (by enumerating the countable family $(\varepsilon_{n,m})_{n,m}$, we can successively choose the $\varepsilon_{n,m}$ such that they are all different) and $x_{\varepsilon_{n,m}} \in K$ such that $\left|\partial^{\alpha_n} u_{\varepsilon_{n,m}}(x_{\varepsilon_{n,m}})\right| > \varepsilon_{n,m}^{-n|\alpha_n|-n}$, for each $n, m \in \mathbb{N}$. Let $x_\varepsilon \in K$ arbitrary if $\varepsilon \in (0,1) \setminus \{\varepsilon_{n,m} : n, m \in \mathbb{N}\}$. By assumption, there exist $k \in \mathbb{R}$, $(p_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $N \in \mathbb{N}$ such that for each $\alpha \in \mathbb{N}^d$ with $|\alpha| \ge N$, $|\partial^\alpha u_\varepsilon(x_\varepsilon)| \le \varepsilon^{p_{|\alpha|}} \le \varepsilon^{-k|\alpha|}$, for small ε. Since $u \in \mathcal{G}(\Omega)$, it follows that there exists $b \in \mathbb{R}$ such that for each $\alpha \in \mathbb{N}^d$, $|\partial^\alpha u_\varepsilon(x_\varepsilon)| \le \varepsilon^{-k|\alpha|-b}$, for small ε. This contradicts the fact that for $n \in \mathbb{N}$ with $n \ge k$ and $n \ge b$, $\lim_m \varepsilon_{n,m} = 0$ and $|\partial^{\alpha_n} u_{\varepsilon_{n,m}}(x_{\varepsilon_{n,m}})| > \varepsilon_{n,m}^{-n|\alpha_n|-n}$, $\forall m \in \mathbb{N}$.

$\Leftarrow$: let $K \subset\subset \Omega$ and $(x_\varepsilon)_\varepsilon \in K^{(0,1)}$. By assumption, there exist $a, b \in \mathbb{R}$ such that for each $\alpha \in \mathbb{N}^d$, $p_{\alpha,K}(u_\varepsilon) \le \varepsilon^{-a|\alpha|-b}$, for small ε. Then, for $k := a+1$ and $p_n := -an-b$, $\lim_n p_n + kn = \infty$ and for each $\alpha \in \mathbb{N}^d$, $|\partial^\alpha u_\varepsilon(x_\varepsilon)| \le p_{\alpha,K}(u_\varepsilon) \le \varepsilon^{p_{|\alpha|}}$, for small ε. $\square$

3 $\mathcal{G}^\infty(\Omega)$ and $\mathcal{G}_{\mathcal{L}_0}(\Omega)$

Our method is based upon a quantitative version of an argument used in [8, Thm. 1.2.3] (cf. also [10, Prop. 1.6] and [17]), which can in fact be traced back to [11].

Proposition 3.1. *Let $K \subset\subset \Omega \subseteq \mathbb{R}^d$. Suppose that there exists $r \in \mathbb{R}^+$ such that for each $x \in K$, there exist d line segments of length r containing x in linearly independent directions that are contained in K. Let $u \in \mathcal{G}(\Omega)$. If for some $k \in \mathbb{N} \setminus \{0\}$, $P_{k,K}(u) > P_{k-1,K}(u)$, then $P_{k,K}^2(u) \le P_{k-1,K}(u)P_{k+1,K}(u)$.*

Proof. Let first $k = 1$. Let $x \in K$. Let $e_1, \dots, e_d \in \mathbb{R}^d$ be linearly independent unit vectors such that the line segments $[x, x + \frac{r}{2}e_j] \subseteq K$. Denote the directional derivative in the direction e_j by ∂_{e_j}. Let $a \in \mathbb{R}$, $a > 0$. For $\varepsilon \in (0,1)$, by Taylor's formula there exist $\theta_\varepsilon \in [0,1]$ such that

$$\partial_{e_j} u_\varepsilon(x) = \varepsilon^{-a} u_\varepsilon(x + \varepsilon^a e_j) - \varepsilon^{-a} u_\varepsilon(x) + \frac{\varepsilon^a}{2} \partial^2_{e_j} u_\varepsilon(x + \varepsilon^a \theta_\varepsilon e_j).$$

Hence for $\varepsilon \le \varepsilon_0$ (where ε_0 does not depend on $x \in K$),

$$\left|\partial_{e_j} u_\varepsilon(x)\right| \le 2\varepsilon^{-a} \sup_{y\in K} |u_\varepsilon(y)| + \varepsilon^a \sup_{y \in K} \left|\partial^2_{e_j} u_\varepsilon(y)\right| \le 2\varepsilon^{-a} p_{0,K}(u_\varepsilon) + \varepsilon^a p_{2,K}(u_\varepsilon).$$

Since $e_1, \dots, e_d$ are linearly independent, we can write $\partial_1, \dots, \partial_d$ as a linear combination (with coefficients independent of ε and x) of $\partial_{e_1}, \dots, \partial_{e_d}$. Thus there exists $C \in \mathbb{R}$ such that $p_{1,K}(u_\varepsilon) \le C\varepsilon^{-a} p_{0,K}(u_\varepsilon) + C\varepsilon^a p_{2,K}(u_\varepsilon)$, and $P_{1,K}(u) \le \max(e^a P_{0,K}(u), e^{-a} P_{2,K}(u))$. Should $P_{2,K}(u) \le P_{0,K}(u)$, then letting $a \to 0$ would yield $P_{1,K}(u) \le P_{0,K}(u)$, contradicting the hypotheses. Hence $P_{2,K}(u) > P_{0,K}(u)$, and we can choose $a > 0$ such that $e^{2a} = P_{2,K}(u)/P_{0,K}(u)$ (since the case $P_{0,K}(u) = 0$ is trivial).
If $k \in \mathbb{N} \setminus \{0\}$ arbitrary, the same reasoning can be applied to all $\partial^\alpha u$ with $|\alpha| = k-1$ instead of u. $\square$

Corollary 3.2. *(cf. [8, Thm. 1.2.3]) Let $K \subset\subset \Omega \subseteq \mathbb{R}^d$. Suppose that there exists $r \in \mathbb{R}^+$ such that for each $x \in K$, there exist d line segments of length r containing x in linearly independent directions that are contained in K. Let $u \in \mathcal{G}(\Omega)$. If for some $k \in \mathbb{N}$, $P_{k,K}(u) = 0$, then $P_{l,K}(u) = 0$, $\forall l \ge k$.*

Proof. If $P_{k+1,K}(u) \ne 0$, then $P_{k+1,K}(u)^2 \le P_{k,K}(u)P_{k+2,K}(u) = 0$ by proposition 3.1, a contradiction. The result follows inductively. $\square$

Proposition 3.3. *Let $K \subset\subset \Omega$ satisfy the hypothesis of proposition 3.1. Let $u \in \mathcal{G}_{\mathcal{L}_0}(K)$. Then $P_{k,K}(u)$ are decreasing in k, and*

$$\mathcal{G}^\infty(K) = \mathcal{G}_{\mathcal{L}_0}(K) = \{u \in \mathcal{G}(\Omega) : P_{k,K}(u) \le P_{0,K}(u), \forall k \in \mathbb{N}\}.$$

In particular, $\mathcal{G}^\infty(K)$ is closed in $\mathcal{G}(\Omega)$.

Proof. Let $u \in \mathcal{G}(\Omega)$. If $P_{k,K}(u)$ are not decreasing in k, then there exists $k \in \mathbb{N} \setminus \{0\}$ such that $P_{k,K}(u) > P_{k-1,K}(u) > 0$ by corollary 3.2. Let $r := P_{k,K}(u)/P_{k-1,K}(u) > 1$. By proposition 3.1, $P_{k+1,K}(u) \geq rP_{k,K}(u)$ (in particular, $P_{k+1,K}(u) > P_{k,K}(u)$). Inductively, $P_{k+n,K}(u) \geq r^n P_{k,K}(u)$, for each $n \in \mathbb{N}$. Thus $\limsup_{n\to\infty} \frac{\ln P_{n+k,K}(u)}{n+k} \geq \limsup_{n\to\infty} \frac{\ln(r^n P_{k,K}(u))}{n+k} = \ln r > 0$, and $u \notin \mathcal{G}_{\mathcal{L}_0}(K)$. In particular, $u \notin \mathcal{G}^\infty(K)$.
The fact that $\mathcal{G}^\infty(K)$ is closed follows by continuity of $P_{k,K}$. $\square$

Theorem 3.4. *$\mathcal{G}^\infty(\Omega) = \mathcal{G}_{\mathcal{L}_0}(\Omega)$ is closed in $\mathcal{G}(\Omega)$. In particular, $\mathcal{G}^\infty(\Omega)$ is not dense in $\mathcal{G}(\Omega)$.*

Proof. $\mathcal{G}^\infty(\Omega) = \bigcap_K \mathcal{G}^\infty(K)$, where K runs over all compact subsets of Ω that are a finite union of d-dimensional cubes parallel with the coordinate axes (hence satisfying the hypothesis of proposition 3.1), and similarly for $\mathcal{G}_{\mathcal{L}_0}(\Omega)$. The conclusions follow from proposition 3.3. $\square$

4 $\mathcal{G}_{\mathcal{L}_a}(\Omega)$, $a > 0$

Proposition 4.1. *Let $K \subset\subset \Omega$ satisfy the hypothesis of proposition 3.1. Let $a \in \mathbb{R}$, $a \geq 1$. Then*

$$\{u \in \mathcal{G}(\Omega) : (\exists c \in \mathbb{R})(P_{k,K}(u) \leq ca^k, \forall k \in \mathbb{N})\} \\ = \{u \in \mathcal{G}(\Omega) : P_{k+1,K}(u) \leq aP_{k,K}(u), \forall k \in \mathbb{N}\}.$$

In particular, this describes a closed subset of $\mathcal{G}(\Omega)$.

Proof. Let $u \in \mathcal{G}(\Omega)$. If $P_{k+1,K}(u) > aP_{k,K}(u)$, for some $k \in \mathbb{N}$, then $P_{k,K}(u) > 0$ by corollary 3.2. Let $r := P_{k+1,K}(u)/P_{k,K}(u) > a$. By proposition 3.1, $P_{k+n,K}(u) \geq r^n P_{k,K}(u)$, for each $n \in \mathbb{N}$. Thus $\limsup_{n\in\mathbb{N}} P_{n,K}(u)/a^n \geq \limsup_{n\in\mathbb{N}} \frac{r^{n-k} P_{k,K}(u)}{a^n} = +\infty$.
The other inclusion is clear. $\square$

Theorem 4.2. *Let $a \in \mathbb{R}$, $a > 0$. Then $\mathcal{G}_{\mathcal{L}_a}(\Omega)$ is not dense in $\mathcal{G}(\Omega)$.*

Proof. $\mathcal{G}_{\mathcal{L}_a}(\Omega) \subseteq \bigcap_K \{u \in \mathcal{G}(\Omega) : (\exists c \in \mathbb{R})(P_{k,K}(u) \leq ce^{ak}, \forall k \in \mathbb{N})\} =: \mathcal{A}$, where K runs over all compact subsets of Ω that are a finite union of d-dimensional cubes parallel with the coordinate axes. The set $\mathcal{A}$ is closed by proposition 4.1 and is a strict subset of $\mathcal{G}(\Omega)$. $\square$

5 Sublinear generalized functions

In order to investigate the density of the algebra of sublinear generalized functions, we start with the following proposition (see also [16, Prop. 4.3.1]):

Proposition 5.1. *Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $\psi(x) = 0$ if $|x| \geq 1$ and $\int_{\mathbb{R}^d} \psi = 1$. Denote by $\psi_\varepsilon(x) := \varepsilon^{-d}\psi(x/\varepsilon)$, for each $\varepsilon \in (0,1)$. If $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$, then $\lim_{n\to\infty}[(u_\varepsilon \star \psi_{\varepsilon^n})_\varepsilon] = u$.*

Proof. Let $n \in \mathbb{N}$ and $K \subset\subset \Omega$. Then $u_\varepsilon \star \psi_{\varepsilon^n}(x) = \int_{|t|\le\varepsilon^n} u_\varepsilon(x-t)\psi_{\varepsilon^n}(t)\,dt$ is well-defined as soon as $d(x, \mathbb{R}^d \setminus \Omega) > \varepsilon^n$. For small ε, $d(K, \mathbb{R} \setminus \Omega) > \varepsilon^n$ and thus $(u_\varepsilon \star \psi_{\varepsilon^n})_{|K}$ can be extended to a $\mathcal{C}^\infty(\Omega)$-function. Independent of the extension, by the mean value theorem,

$$p_{k,K}(u_\varepsilon \star \psi_{\varepsilon^n} - u_\varepsilon) = \sup_{x\in K, |\alpha|=k} |(\partial^\alpha u_\varepsilon) \star \psi_{\varepsilon^n}(x) - \partial^\alpha u_\varepsilon(x)|$$
$$= \sup_{x\in K,|\alpha|=k} \left| \int_{|t|\le\varepsilon^n} (\partial^\alpha u_\varepsilon(x-t) - \partial^\alpha u_\varepsilon(x))\psi_{\varepsilon^n}(t)\,dt \right| \le \varepsilon^n p_{k+1,K+r}(u_\varepsilon) \int_{\mathbb{R}^d} |\psi|$$

for small ε, where $r > 0$ ($r \in \mathbb{R}$) such that $K + r = \{x \in \mathbb{R}^d : d(x, K) \le r\} \subset\subset \Omega$. □

Proposition 5.2. *Let $\mathcal{A}$ be the set of all $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_c(\Omega)$ for which*

$$(\exists N \in \mathbb{N})(\forall K \subset\subset \Omega)(\forall k \in \mathbb{N})(p_{k,K}(u_\varepsilon) \le \varepsilon^{-Nk-N},\ \textit{for small } \varepsilon).$$

Then $\mathcal{A}$ is dense in $\mathcal{G}(\Omega)$.

Proof. Let $u \in \mathcal{G}_c(\Omega)$. Then there exists a representative $(u_\varepsilon)_\varepsilon$ of u and $L \subset\subset \Omega$ such that $\operatorname{supp} u_\varepsilon \subseteq L$, for each ε. For each $K \subset\subset \Omega$ and $k \in \mathbb{N}$,

$$p_{k,K}(u_\varepsilon \star \psi_{\varepsilon^n}) = \sup_{x\in K,|\alpha|=k} |u_\varepsilon \star \partial^\alpha(\psi_{\varepsilon^n})|$$
$$\le \varepsilon^{-nk} \sup_{x\in L} |u_\varepsilon(x)| \max_{|\alpha|=k} \int_{\mathbb{R}^d} |\partial^\alpha \psi| \le \varepsilon^{-nk-1} \sup_{x\in L} |u_\varepsilon(x)|\,,$$

for small ε. Thus $[(u_\varepsilon \star \psi_{\varepsilon^n})_\varepsilon] \in \mathcal{A}$. By proposition 5.1, $\mathcal{A}$ is dense in $\mathcal{G}_c(\Omega)$. Further, $\mathcal{G}_c(\Omega)$ is dense in $\mathcal{G}(\Omega)$ (for $u \in \mathcal{G}(\Omega)$, $u = \lim_{n\to\infty} u\chi_n$, where $\chi_n \in \mathcal{D}(\Omega)$ with $\chi_n(x) = 1$, $\forall x \in K_n$, where $(K_n)_{n\in\mathbb{N}}$ is a compact exhaustion of Ω). □

Theorem 5.3. *The subalgebra of sublinear generalized functions is dense in $\mathcal{G}(\Omega)$.*

Proof. With the notations of proposition 5.2, $\mathcal{A} \subseteq \{u \in \mathcal{G}(\Omega) : u \text{ is sublinear}\}$. □

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